EXAM GROUP THEORY,
January 30th, 2020, 8:30am–11:30am,
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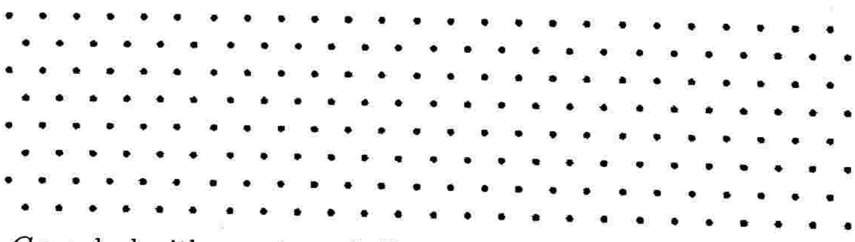
Put your name on every sheet of paper you hand in. Please provide complete arguments for each of your answers. The exam consists of 4 questions. You can score up to 9 points for each question, and you obtain 4 points for free.
In this way you will score in total between 4 and 40 points.

(1) Consider the permutations $\sigma = (1\ 2\ 3\ 4)(2\ 3\ 4\ 5)(4\ 5\ 6\ 7)$ and $\tau = (1\ 2\ 3\ 4)(2\ 3\ 4\ 5)(6\ 7)$ in S_7.
  (a) [2 points.] Write $\tau = \pi^3$ for some $\pi \in S_7$.
  (b) [1 point.] If $\pi \in S_7$ is a k-cycle with $k \in \{2, 4, 5, 7\}$, show that π^3 is a k-cycle as well.
  (c) [1 point.] If $\pi = (a_1\, a_2\, a_3\, a_4\, a_5\, a_6) \in S_7$ is a 6-cycle, show that π^3 is a product of three disjoint 2-cycles.
  (d) [2 points.] Show that σ cannot be written as a third power in S_7.
  (e) [2 points.] Prove that elements of order 3 in S_7 can not be expressed as the third power of any element in S_7.
  (f) [1 point.] Contrary to (e), find $\gamma \in S_9$ of order 3 such that γ is a third power in S_9.

(2) Consider the groups $G_1 = (\mathbb{Z}/100\mathbb{Z})^\times$ and $G_2 = (\mathbb{Z}/110\mathbb{Z})^\times$ and $G_3 = (\mathbb{Z}/132\mathbb{Z})^\times$.
  (a) [3 points.] Show that these three groups have the same number of elements.
  (b) [2 points.] Show that G_1 and G_3 are not isomorphic.
  (c) [3 points.] Show that G_1 and G_2 are isomorphic (hint: probably the shortest way to do this, is to use both the Chinese Remainder Theorem and the theory of elementary divisors).
  (d) [1 point.] Give a non-commutative group with the same number of elements as G_1.

(3) This problem considers the symmetry group of the so-called "hexagonal lattice". We begin by introducing this group, as a subgroup of the group of isometries of the Euclidean plane. Put $v_1 = \binom{1}{0} \in \mathbb{R}^2$ and $v_2 = \binom{-\frac{1}{2}}{\frac{1}{2}\sqrt{3}} \in \mathbb{R}^2$. Take $L = \mathbb{Z} \cdot v_1 + \mathbb{Z} \cdot v_2$, it looks like:



The group G we deal with, consists of all isometries of the plane which map L to L:

$$G := \{\iota \in \mathrm{Isom}(\mathbb{R}^2) \mid \iota(L) = L\}.$$

To describe the elements of G, we first consider the ones that moreover map $\binom{0}{0}$ to itself. These form a subgroup $H \subset G$, with $H \cong D_6$ (the dihedral group of order 12).
Another subgroup of G is described as follows. For $v \in L$ we denote by τ_v the translation over v, so $\tau_v(x) = x + v$ (any $x \in \mathbb{R}^2$). Then

$$T := \{\tau_v \mid v \in L\}$$

is a subgroup of G. Any $\iota \in G$ can be written as $\iota = \tau_v \circ \sigma$ for some $\tau_v \in T$ and some $\sigma \in H$. (You may accept these statements without proving them.)

(a) [2 points.] Show that if $\tau_v, \tau_w \in T$ and $\sigma_1, \sigma_2 \in H$ then $(\tau_v\sigma_1)\circ(\tau_w\sigma_2) = \tau_{v+\sigma_1(w)}\sigma_1\sigma_2$.
(b) [2 points.] Show that if $\tau_v, \tau_w \in T$ and $\sigma \in H$ then $(\tau_w\sigma)\tau_v(\tau_w\sigma)^{-1} = \tau_{\sigma(w)}$.
(c) [1 point.] Prove that T is normal in G.
(d) [2 points.] Prove that $G/T \cong D_6$ (the dihedral group of order 12).
(e) [2 points.] Recall $v_1 = \binom{1}{0} \in \mathbb{R}^2$; by definition $\tau_{v_1} \in T$. Note that $-id \in H$. Find the conjugacy class in G of $\tau_{v_1} \circ (-id) \in G$.

(4) In this (final) exercise we apply the "orbit-counting formula" in order to calculate the number of monomials of a given degree. Having variables $x_1, x_2, \ldots, x_n$, a monomial of degree d is an expression $x_1^{j_1}x_2^{j_2}\cdot\ldots\cdot x_n^{j_n}$ with all $j_k \in \mathbb{Z}_{\geq 0}$ and $\sum j_k = d$. For example, the monomials of degree 3 in the variables x, y are x^3, x^2y, xy^2, y^3.

One obtains all monomials of degree d in the n variables as follows. Denote by X the set of all functions $f\colon \{1, 2, \ldots, d\} \to \{x_1, x_2, \ldots x_n\}$. Given $f \in X$, one obtains a monomial of degree d by taking the product $f(1)\cdot f(2)\cdot\ldots\cdot f(d)$. Note that given a permutation $\sigma \in S_d$ and $f \in X$, also the composition $f\circ\sigma \in X$. This defines $S_d \times X \to X \quad (\sigma, f) \mapsto \sigma * f := f\circ\sigma$.

(a) [2 points.] Show that $\sigma * f$ defines an action of S_d on X.
(b) [1 point.] Show that $\#X = n^d$
(c) [2 points.] Show that $f, g \in X$ yield the same monomial, if and only if the orbit $S_d * f$ of f equals the orbit $S_d * g$ of g.
(d) [2 points.] For $\tau \in S_d$ a 2-cycle, show $\#\mathrm{Stab}(\tau) = n^{d-1}$.
(e) [2 points.] Take $d = 4$, and use the "orbit-counting formula" to determine the number of monomials of degree 4 in n variables.